

Diffusion in a potential landscape with stochastic resetting

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The steady state of a Brownian particle diffusing in an arbitrary potential under the stochastic resetting mechanism has been studied. We show that there are different classes of nonequilibrium steady states depending on the nature of the potential. In the stable potential landscape, the system attains a well defined steady state however existence of the steady state for the unstable landscape is constrained. We have also investigated the transient properties of the propagator towards the steady state under the stochastic resetting mechanism. Finally, we have done numerical simulations to verify our analytical results.

I. INTRODUCTION

Diffusion with stochastic resetting is considered to be a natural framework for the study of intermittent search processes [1, 2]. The simplest question of finding a lost object being a key or a car or an offender is one of the central quests of the discipline. The search processes related to resetting are realized in diverse fields such as biochemistry (where a signaling molecule is reset back to a receptor protein in the membrane depending on the concentration of certain molecules in the vicinity) [3], computer network (to find an element in a sorted and pivoted array) [4], ecology (Capuchin monkeys, known for long term memory, foraging a territory with palm nuts) [5] and microbiology [6]. In addition, this mechanism was considered to compute the stationary distribution of variant models of population growth where the population is stochastically reset to some higher or lower values leading to a power law growth [7, 8]. Also, there have been interests to study the continuous time random walk where the both the position and the waiting time are chosen from certain distributions in the presence of resetting [9].

‘Stochastic resetting’ is a mechanism where a Brownian particle is stochastically reset to its initial position at a constant rate thus driving the system away from any equilibrium state [10–13]. It is thus a simple mechanism to generate a non equilibrium stationary state. In such states probability currents are non-zero and detailed balance does not hold naturally. Of late, the implication of the stochastic reset has been studied in the one dimensional reaction-diffusion systems where a finite reset rate leads to an unique non equilibrium stationary state [14]. The interface growth models described by Kardar-Parisi-Zhang and Edwards-Wilkinson equations also exhibit nonequilibrium stationary states with non-Gaussian interface fluctuations when the interface stochastically resets to a fixed initial profile at a constant rate [15]. In this backdrop, a natural question to ask would be: is the non equilibrium stationary state generic to any dynamics subjected to stochastic resetting. The primary goal of this paper is to address this question. To gain insights, one considers model systems which are simple enough though addresses the basic moral. In this paper, we consider a simple model of a Brownian particle diffusing in an arbitrary potential landscape in the presence of stochastic resetting. It is obvious that for a bounded case, even without reset one gets a steady state around the minimum of the potential. But when the equilib-

rium point of the potential differs from the reset point, two mechanisms compete with each other and finally reaches a steady state which shows certain generic behavior. On the other hand, for a particle diffusing in an unbounded potential, there exists no steady state at all in the absence of resetting. We propose to invoke stochastic resetting to retrieve the steady state. However, this behavior is not universal and rather puts a general constraint on the nature of the potential. We derive the conditions that ensures the steady state in the case of an unbounded potential.

The paper is structured as follows. In the following section, we introduce the model and the resetting dynamics. In Sec. III, we obtain exact steady state distribution $P_{st}(x|x_0)$ for two representative choices of the potential $V(x)$, namely, (i) $V(x) \sim \mu|x|$, and (ii) $V(x) \sim \mu x^2$. The positive and negative sign of μ describe bounded and unbounded potential respectively. We also derive the conditions to obtain an unique steady state for an arbitrary potential landscape. In Sec. IV, we investigate the transient behavior of the propagator in the presence of the resetting. We conclude with a summary and future directions in Sec. V.

II. THE MODEL

Consider a single particle undergoing diffusion in one dimension in presence of an external potential $V(x)$:

$$\frac{dx}{dt} = -V'(x) + \eta(t), \quad (1)$$

where $\eta(t)$ is a Gaussian white noise with

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = 2D\delta(t - t'), \quad (2)$$

D being the diffusion constant and the viscosity of the medium has been scaled to unity for brevity. Here, angular brackets denote averaging over noise realizations. The initial condition is

$$x(0) = x_0, \quad (3)$$

where $x_0 \in (0, \infty]$. We now introduce the ‘Stochastic resetting’ mechanism by which the particle returns to its initial location at a constant rate r . To elaborate, in a small time Δt , the particle is reset to the initial position $x = x_0$ with probability $r\Delta t$, while with the complementary probability $1 - r\Delta t$, the particle dynamics follows Eq. (1).

III. STEADY STATE DISTRIBUTION

Let $P(x, t|x_0)$ be the probability to find the particle at position x at time t , given that it was at x_0 at time $t = 0$. From the dynamical rules for the evolution of the particle given in the preceding section, it follows that

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} + \frac{\partial[V'(x)P]}{\partial x} - rP + r\delta(x - x_0), \quad (4)$$

with the initial condition $P(x, 0|x_0) = \delta(x - x_0)$. Here, the third and fourth terms on the right hand side (rhs) account for the resetting events, denoting the negative probability flux $-rP$ from each point x and a corresponding positive probability flux into $x = x_0$. The steady state solution $P_{\text{st}}(x|x_0)$ satisfies

$$0 = D \frac{d^2 P_{\text{st}}}{dx^2} + \frac{d[V'(x)P_{\text{st}}]}{dx} - rP_{\text{st}} + r\delta(x - x_0). \quad (5)$$

In the following section, we have investigated steady state distributions for various bounded and unbounded potential landscapes. In particular, we have studied for two representative choices of the potential $V(x)$, namely, (i) $V(x) \sim \mu|x|$, and (ii) $V(x) \sim \mu x^2$.

A. The case of a mod potential

We first consider the case of a mod potential. This potential is centred either around its minimum or the maximum at 0. The reset location is at $x_0 \neq 0$. The nature of the potential allows us to identify three regions in x , namely, region I ($x > x_0$), region II ($0 < x < x_0$), and region III ($x < 0$). To find the steady state, we solve Eq. (5) in each region and require that the solutions are continuous at $x = x_0$ and $x = 0$. Though, the derivatives are discontinuous and it can be seen by integrating Eq. (5) over an infinitesimal region around $x = x_0$,

$$\left. \frac{dP_{\text{st}}^I(x|x_0)}{dx} \right|_{x=x_0} - \left. \frac{dP_{\text{st}}^{II}(x|x_0)}{dx} \right|_{x=x_0} = -\frac{r}{D}, \quad (6)$$

This discontinuity does not depend on μ indicating the robustness of ‘kinks’ present at x_0 irrespective of potential landscapes. On the other hand, while integrating Eq. (5) over an infinitesimal region around $x = 0$, we find that

$$\left. \frac{dP_{\text{st}}^{II}(x|x_0)}{dx} \right|_{x=0} - \left. \frac{dP_{\text{st}}^{III}(x|x_0)}{dx} \right|_{x=0} = \mp \frac{2\mu}{D} P_{\text{st}}^{II}(x|x_0) \Big|_{x=0}, \quad (7)$$

in which minus and plus signs are for the bounded and the unbounded case respectively. In the following subsections, we consider these two cases respectively.

1. Bounded potential: $V(x) = \mu|x|$, $\mu > 0$

We first consider the case where $\mu > 0$. Using the Eq. (6) and Eq. (7), we obtain the steady state solutions given by,

$$\begin{aligned} P_{\text{st}}^I(x|x_0) &= \frac{r}{\sqrt{\mu^2 + 4Dr}} e^{-m_2 x_0} e^{m_2 x} \\ &\quad + \frac{\mu r}{\sqrt{\mu^2 + 4Dr}(\sqrt{\mu^2 + 4Dr} - \mu)} e^{-m_1 x_0} e^{m_2 x}, \\ P_{\text{st}}^{II}(x|x_0) &= \frac{r}{\sqrt{\mu^2 + 4Dr}} e^{-m_1 x_0} e^{m_1 x} \\ &\quad + \frac{\mu r}{\sqrt{\mu^2 + 4Dr}(\sqrt{\mu^2 + 4Dr} - \mu)} e^{-m_1 x_0} e^{m_2 x}, \\ P_{\text{st}}^{III}(x|x_0) &= \frac{r}{\sqrt{\mu^2 + 4Dr} - \mu} e^{-m_1 x_0} e^{-m_2 x}, \end{aligned} \quad (8)$$

where

$$m_1 = \frac{-\mu + \sqrt{\mu^2 + 4Dr}}{2D}, \quad m_2 = -\frac{\mu + \sqrt{\mu^2 + 4Dr}}{2D}. \quad (9)$$

Fig. 1 shows a comparison between simulations and theory for steady state Eq. (8) demonstrating a very good agreement. From the solution, it is evident that $P_{\text{st}}(x|x_0)$ exhibits two cusps where its derivatives are discontinuous, namely, (i) at the resetting location $x = x_0$, and (ii) at $x = 0$, the point at which the potential $V(x)$ has discontinuous derivatives.

2. Unbounded potential: $V(x) = \mu|x|$, $\mu < 0$

The steady state solutions for the unbounded case when $\mu < 0$ are given in the following,

$$\begin{aligned} P_{\text{st}}^I(x|x_0) &= \frac{r}{\sqrt{\mu^2 + 4Dr}} e^{m_1 x_0} e^{-m_1 x} \\ &\quad - \frac{\mu r}{\sqrt{\mu^2 + 4Dr}(\sqrt{\mu^2 + 4Dr} + \mu)} e^{m_2 x_0} e^{-m_1 x}, \\ P_{\text{st}}^{II}(x|x_0) &= \frac{r}{\sqrt{\mu^2 + 4Dr}} e^{m_2 x_0} e^{-m_2 x} \\ &\quad - \frac{\mu r}{\sqrt{\mu^2 + 4Dr}(\sqrt{\mu^2 + 4Dr} + \mu)} e^{m_2 x_0} e^{-m_1 x}, \\ P_{\text{st}}^{III}(x|x_0) &= \frac{r}{\sqrt{\mu^2 + 4Dr} + \mu} e^{m_2 x_0} e^{m_1 x}, \end{aligned} \quad (10)$$

where m_1, m_2 are given by Eq. (9).

Fig. 2 shows a comparison between simulations and theory for steady state Eq. (10) demonstrating a very good agreement. Again $P_{\text{st}}(x|x_0)$ exhibits two cusps where its derivatives are discontinuous, namely, (i) at the resetting location $x = x_0$, and (ii) at $x = 0$, the point at which the potential $V(x)$ has discontinuous derivatives.

B. The case of a quadratic potential

We now consider the case of a harmonic potential centred around 0 which is either its minimum or the maximum. As

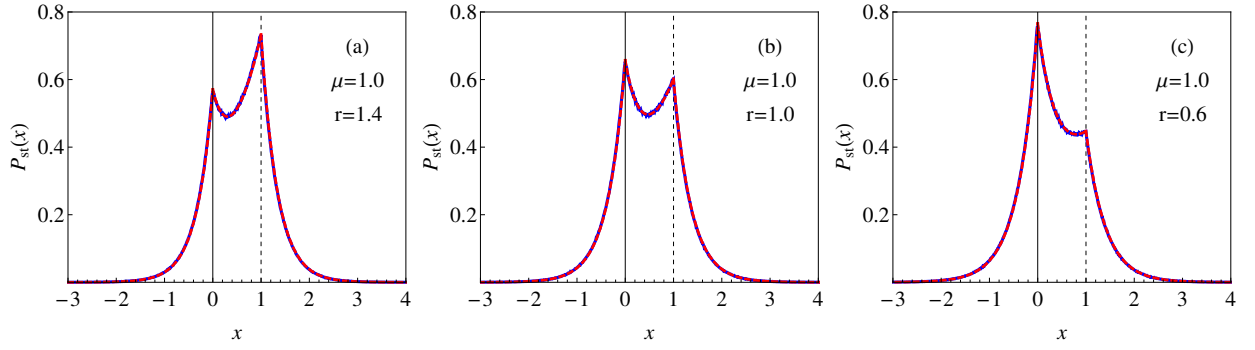


FIG. 1. (Color online) Stationary distribution $P_{\text{st}}(x|x_0)$ for the bounded potential $V(x) = \mu|x|$, with $\mu > 0$. We choose $D = 0.5, x_0 = 1.0, \mu = 1.0$ while vary r . The (red) dashed line plots the analytical result for $P_{\text{st}}(x)$, while the (blue) points are numerical simulation results. Also, The vertical solid and dashed lines indicate the location of the stable minimum of the bounded potential and the reset point x_0 respectively. The motion of the peak is also clear from the figure.

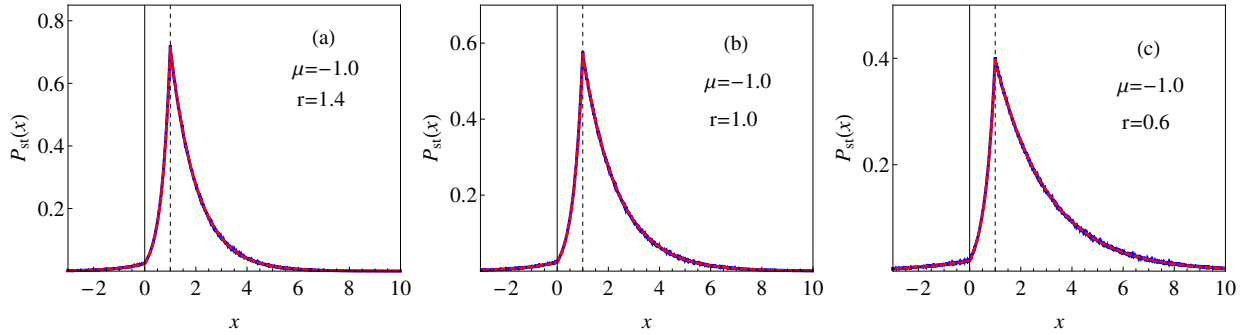


FIG. 2. (Color online) Stationary distribution $P_{\text{st}}(x|x_0)$ for the unbounded potential $V(x) = \mu|x|$, with $\mu < 0$. We choose $D = 0.5, x_0 = 1.0, \mu = -1.0$ while vary r . The (red) dashed line plots the analytical result for $P_{\text{st}}(x)$, while the (blue) points are numerical simulation results. Also, the vertical solid and dashed lines indicate the location of the unstable maximum of the unbounded potential and the reset point x_0 respectively.

before, reset takes place at x_0 . One can again identify two regions in x , namely, region I ($x > x_0$), and region II ($x < x_0$). We solve Eq. (5) in each region and use the fact that the solutions are continuous at $x = x_0$ while the derivatives are not. This can be seen by integrating Eq. (5) over an infinitesimal region around $x = x_0$ where one finds

$$\left. \frac{dP_{\text{st}}^I(x|x_0)}{dx} \right|_{x=x_0} - \left. \frac{dP_{\text{st}}^{II}(x|x_0)}{dx} \right|_{x=x_0} = -\frac{r}{D}. \quad (11)$$

This is consistent with the fact mentioned in Eq. (6). Similar to the last section, in the following we derive the steady state solutions for both the stable and the unstable landscape.

1. Bounded potential: $V(x) = (\mu/2)x^2$

We first consider the case where $\mu > 0$ and this is the case of a bounded harmonic potential. The solutions are then given by

$$P_{\text{st}}^I(x|x_0) = c_1 e^{-\frac{\mu}{2D}x^2} H\left(-\frac{r}{\mu}, \sqrt{\frac{\mu}{2D}}x\right) + c_2 e^{-\frac{\mu}{2D}x^2} {}_1F_1\left(\frac{r}{2\mu}; \frac{1}{2}; \frac{\mu}{2D}x^2\right),$$

$$P_{\text{st}}^{II}(x|x_0) = c_3 e^{-\frac{\mu}{2D}x^2} H\left(-\frac{r}{\mu}, \sqrt{\frac{\mu}{2D}}x\right) + c_4 e^{-\frac{\mu}{2D}x^2} {}_1F_1\left(\frac{r}{2\mu}; \frac{1}{2}; \frac{\mu}{2D}x^2\right), \quad (12)$$

where $H(-n, x)$ is the Hermite polynomial of negative order n , and ${}_1F_1(a; b; x)$ is the Kummer confluent hypergeometric function. We note that $H(-n, \sqrt{\mu}x)$ converges as x^{-n} when $x \rightarrow \infty$ but diverges as $x^{n-1}e^{\mu x^2}$ when $x \rightarrow -\infty$. But ${}_1F_1(a; b; \mu x^2)$ is even in x and diverges as $e^{\mu x^2} x^{a-b}$ when $x \rightarrow \pm\infty$. However, these functions are multiplied with $e^{-\mu x^2}$ and then the exponentials cancel each other which makes the additional algebraic form important at the asymptotic limits. This results in two distinct situations namely $r \geq \mu$ and $r < \mu$. In the first case, one needs to choose $c_2 = 0$ for the convergence of the steady state. However, in the second case, one can show that it is not necessary to choose $c_2 = 0$, rather there are infinite choices for c_2 and for each, c_1 will be automatically determined by the normalization condition. In this paper, we choose $c_2 = 0$ to maintain an identical structure between the two cases.

For further analysis, let us choose $D = 1/2$, without loss of generality. It will prove useful to define the following quanti-

ties:

$$z_1(r, \mu, x_0) \equiv \sqrt{\mu} x_0 H\left(-\frac{r}{\mu}, \sqrt{\mu} x_0\right) {}_1F_1\left(1 + \frac{r}{2\mu}; \frac{3}{2}; \mu x_0^2\right) + H\left(-1 - \frac{r}{\mu}, \sqrt{\mu} x_0\right) {}_1F_1\left(\frac{r}{2\mu}; \frac{1}{2}; \mu x_0^2\right), \quad (13)$$

$$a_1(r, \mu, x_0) \equiv \sqrt{\mu} e^{\mu x_0^2} {}_1F_1\left(\frac{r}{2\mu}; \frac{1}{2}; \mu x_0^2\right), \quad (14)$$

$$b_1(r, \mu, x_0) \equiv \sqrt{\mu} e^{\mu x_0^2} H\left(-\frac{r}{\mu}, \sqrt{\mu} x_0\right). \quad (15)$$

Using these definitions and from Eq. (11), we get

$$c_3 = c_1 - \frac{a_1(r, \mu, x_0)}{z_1(r, \mu, x_0)}, \quad (16)$$

$$c_4 = \frac{b_1(r, \mu, x_0)}{z_1(r, \mu, x_0)}. \quad (17)$$

Thus, c_4 is independent of c_1 , while c_3 depends on c_1 and can be evaluated once c_1 is found from the normalization condition:

$$\int_{-\infty}^{x_0} dx P_{\text{st}}^I(x|x_0) + \int_{x_0}^{\infty} dx P_{\text{st}}^I(x|x_0) = 1. \quad (18)$$

That said, one obtains the full steady state solutions from Eq. (12).

Fig. 3 shows a comparison between simulations and theory for steady state Eq. (12) demonstrating a very good agreement. We note that, there is only cusp at the reset point $x = x_0$. When r is large compared to μ , the distribution is peaked around $x = x_0$ with a non-Gaussian form. However, when μ is much greater than r , we get a distribution peaked around the minimum of the potential. In between, the peak moves between x_0 and the minimum. This generic feature of the distribution is clear from Fig. 3.

2. Unbounded potential: $V(x) = -(\mu/2)x^2$

We proceed further with a similar analysis in the case of the unbounded harmonic potential and the solutions are given by

$$P_{\text{st}}^I(x|x_0) = d_1 H\left(-1 - \frac{r}{\mu}, \sqrt{\frac{\mu}{2D}}x\right) \\ P_{\text{st}}^{II}(x|x_0) = d_3 H\left(-1 - \frac{r}{\mu}, \sqrt{\frac{\mu}{2D}}x\right) + d_4 {}_1F_1\left(\frac{1}{2} + \frac{r}{2\mu}; \frac{1}{2}; \frac{\mu}{2D}x^2\right), \quad (19)$$

where $H(-n, x)$ is the Hermite polynomial of negative order n , and ${}_1F_1(a; b; x)$ is the Kummer confluent hypergeometric function same as before. Choosing $D = 1/2$ and using the boundary conditions Eq. (11), one obtains

$$d_3 = d_1 - \frac{a_2(r, \mu, x_0)}{z_2(r, \mu, x_0)}, \quad (20)$$

$$d_4 = \frac{b_2(r, \mu, x_0)}{z_2(r, \mu, x_0)}, \quad (21)$$

where

$$z_2(r, \mu, x_0) \equiv (r + \mu) \left[\sqrt{\mu} x_0 H\left(-1 - \frac{r}{\mu}, \sqrt{\mu} x_0\right) {}_1F_1\left(\frac{3}{2} + \frac{r}{2\mu}; \frac{3}{2}; \mu x_0^2\right) + H\left(-2 - \frac{r}{\mu}, \sqrt{\mu} x_0\right) {}_1F_1\left(\frac{1}{2} + \frac{r}{2\mu}; \frac{1}{2}; \mu x_0^2\right) \right], \quad (22)$$

$$a_2(r, \mu, x_0) \equiv r \sqrt{\mu} {}_1F_1\left(\frac{1}{2} + \frac{r}{2\mu}; \frac{1}{2}; \mu x_0^2\right), \quad (23)$$

$$b_2(r, \mu, x_0) \equiv r \sqrt{\mu} H\left(-1 - \frac{r}{\mu}, \sqrt{\mu} x_0\right). \quad (24)$$

Then d_1 can be found using the normalization condition Eq. (18) as before and the solutions are deduced from Eq. (19).

Fig. 4 shows a comparison between simulations and theory for steady state Eq. (19), demonstrating a very good agreement. We note that, there is only cusp at the reset point $x = x_0$. When r is large compared to μ , the distribution is peaked around $x = x_0$ with a non-Gaussian form. However, when μ is much greater than r , the peak does not move unlike the case of the bounded potential. Nevertheless, in this limit, the system takes longer time to reach the steady state with a peak well set at $x = x_0$ indicating a fat tailed distribution at large x [19]. We refer to the Fig. 4 which characterizes this generic feature.

C. General $V(x)$: Possible Steady States

We generalize our discussion for arbitrary potential that has a form $V(x) = \mu|x|^\delta$. When $\mu > 0$, that is the potential is stable with minimum at $x = x_{\min}$, one will always achieve the steady state around x_{\min} irrespective of the resetting. Nevertheless, resetting will invoke the non differentiability in the steady state resulting in a cusp at the reset point $x_0 \neq x_{\min}$. Here one can talk about two extreme limits: one is when the strength of the potential is much greater than the reset rate and one expects a steady state solution of form $\sim e^{-V(x)}$ centred around x_{\min} with a small but non vanishing cusp at x_0 Eq. (6). In the other limit, when reset rate dominates the potential strength, one finds a non Gaussian form around x_0 . However, in between, the peak of the steady state moves from x_{\min} to x_0 as one varies r but keeping μ fixed. This is a generic feature that can be seen for any δ .

Now consider the case when $\mu < 0$. There is no stable minimum of the potential, hence no steady state since the particle escapes to infinity in the absence of stochastic resetting. However, we notice that one can have a steady state when resetting is introduced under certain conditions which we discuss in the following. We can find a steady state if and only if $V(x)$ is such that the particle starting from x_0 does not escape to infinity at a finite time in the absence of resetting. Note that the escape time is given by $t_{\text{esc}} = -\int_{x_0}^{\infty} [V'(x)]^{-1} dx = [x_0^{\delta-2}(\delta-2)\delta\mu]^{-1}$ for $\delta > 2$. On the other hand, the waiting time distribution for resetting is given by Poisson distribution namely $re^{-r\tau}$, with the average time between two resets is simply given by $t_{\text{reset}} = 1/r$, which is always finite. It is then obvious that if $t_{\text{esc}} < t_{\text{reset}}$ the particle always escapes

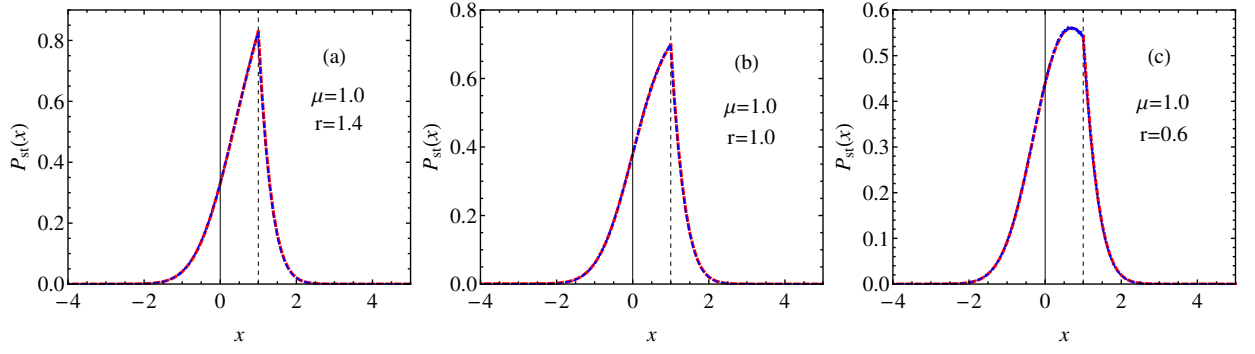


FIG. 3. (Color online) Stationary distribution $P_{\text{st}}(x)$ for the potential $V(x) = (\mu/2)x^2$, with $\mu > 0$. We choose $D = 0.5$, $x_0 = 1.0$, $\mu = 1.0$ while vary r . The (red) dashed line plots the analytical result for $P_{\text{st}}(x)$, while the (blue) points are numerical simulation results. The vertical solid and dashed lines indicate the location of the stable minimum of the bounded potential and the reset point x_0 respectively. The motion of the peak is also clear from the figure.

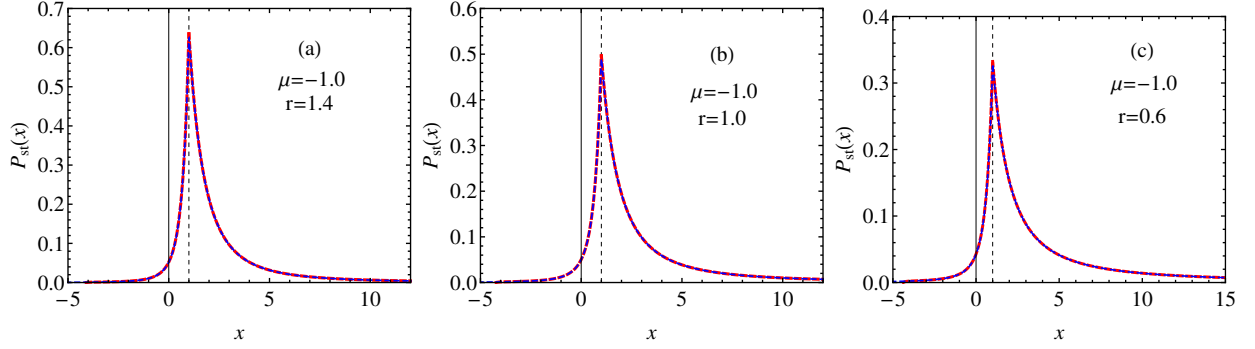


FIG. 4. (Color online) Stationary distribution $P_{\text{st}}(x)$ for the potential $V(x) = (\mu/2)x^2$, with $\mu < 0$. We choose $D = 0.5$, $x_0 = 1.0$, $\mu = -1.0$ while vary r . The (red) dashed line plots the analytical result for $P_{\text{st}}(x)$, while the (blue) points are numerical simulation results. The vertical solid and dashed lines indicate the location of the unstable maximum of the unbounded potential and the reset point x_0 respectively.

and there is no steady state. However, one indeed achieves a steady state if $t_{\text{esc}} > t_{\text{reset}}$ even for $\delta > 2$. This can be realized by increasing the reset rate so that it gets reset promptly before escaping. On the contrary, for $\delta \leq 2$, one finds $t_{\text{esc}} \rightarrow \infty$, thus always maintaining a steady state through resetting. We have discussed the cases of $\delta = 2$ (harmonic) and $\delta = 1$ (mod) for both positive and negative μ in great details. For positive μ , the steady states and the motion of the peak as well is followed from the Fig. 1(b)-(c), Fig. 3(b)-(c). But for negative μ , the peak is always set at x_0 indicating the fact that the steady state is solely due to the reset mechanism. This is realized from Fig. 2, Fig. 4.

IV. RELAXATION TO THE STEADY STATE

In this section, we investigate the transient behavior of the stochastic resetting mechanism. We recall that the particle starts at $x = x_0$ at $t = 0$ and finally attains a steady state either at $x = x_0$ or $x = x_{\text{min}}$ as $t \rightarrow \infty$ depending on the potential landscape. In between, the particle position PDF shows rich behavior which can be quantified by studying the relaxation

dynamics of the propagator. We first recall Eq. (IV)

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} + \frac{\partial [V'(x)P]}{\partial x} - rP + r\delta(x - x_0), \quad (25)$$

with the boundary conditions $P(x \rightarrow \pm\infty, t) = 0$ and the initial condition $P(x, t=0) = \delta(x - x_0)$. Now to characterize the transient states, one has to solve Eq. (IV) for the time dependent propagator. To do this, we first separate $P(x, t) = f(x) + b(t, x)$ where, $f(x)$ gives the steady state solution and $b(t, x)$ describes the relaxation towards it. As a representative case, we choose the free diffusion with no potential. The steady state solution $f(x)$ then satisfies the simple equation Eq. (5) with the boundary conditions $f(x \rightarrow \pm\infty) = 0$,

$$Df''(x) - rf(x) + r\delta(x - x_0) = 0, \quad (26)$$

and this gives rise to the solution

$$f(x) = \frac{\alpha}{2} \exp[-\alpha|x - x_0|], \quad (27)$$

where $\alpha = \sqrt{r/D}$ is an inverse length scale denoting to the typical distance diffused by the particle between the resets [10]. The time dependent part is given by

$$\partial_t b(t, x) = D\partial_x^2 b(t, x) - rb(t, x), \quad (28)$$

with the boundary conditions $b(t, x \rightarrow \pm\infty) = 0$ and $b(t \rightarrow \infty, x) \rightarrow 0$. The initial condition is given by $b_0(x) \equiv b(t=0, x) = P(x, 0) - f(x)$. This results in the complete form of the relaxation term given by

$$b(t, x) = e^{-rt} \frac{\exp\left[-\frac{(x-x_0)^2}{4Dt}\right]}{\sqrt{4\pi Dt}} - \frac{\alpha}{2} \cosh\left[-\alpha(x_0 - x)\right] + \frac{\alpha}{4} \exp\left[-\alpha(x_0 - x)\right] \operatorname{erf}\left[\frac{x - x_0 + 2Dt\alpha}{\sqrt{4\pi Dt}}\right] + \frac{\alpha}{4} \exp\left[\alpha(x_0 - x)\right] \operatorname{erf}\left[\frac{-x + x_0 + 2Dt\alpha}{\sqrt{4\pi Dt}}\right]. \quad (29)$$

Now, Eq. (27) and Eq. (29) constitute the full propagator. We refer to the top panel of Fig. 5 which specifies the relaxation for this particular case. A similar analysis can also be made for a Brownian particle diffusing in a potential in the presence of resetting. For instance, we analyze the case of a bounded harmonic potential $V(x) = (\mu/2)x^2$ with minimum $x_{\min} = 0$ while the reset point is at $x_0 \neq 0$. This gives rise to a competition between the potential and the reset mechanism thus reaching a steady state as discussed in Sec. III C. The bottom panel of Fig. 5 characterizes the transient behavior with respect to t for $\mu = 1.0$, $r = 0.6$. We also notice that the steady state achieved at the end is identical with that obtained in Fig. 3(c).

V. SUMMARY

In this work, we have considered a Brownian particle diffusing in an arbitrary potential landscape in the presence of the stochastic resetting mechanism. We have investigated the steady state properties of the position distribution of the particle for two representative choices of the potential namely the mod and the harmonic potential. It has been shown that the steady states have distinct differences depending on the nature of the potential. We also derive the conditions for the existence of the steady state for any potential landscape of higher order. Also we have realized the transient behavior of the propagator approaching to the steady state. We have studied two representative cases in this context though the extension to higher order potential does not offer more physical insights.

Furthermore, resetting has been found to have a profound consequence on the first passage properties of a diffusing particle. In recent times, there have been extensive studies on this to have a discreet idea not only restricted to one dimension but to higher dimension as well [16]. Consequently, the study of two observables namely the local time and the occupation time turns out to be very useful to understand the mechanism near the reset point. Local time measures the time that the process visits a reference point (which is basically the reset point) while the residence time or the occupation time measures the time that the process stays above that point [17, 18]. These observables show rich behavior when the resetting dynamics is combined [19]. There are lot of open questions in the context of stochastic resetting mechanism.

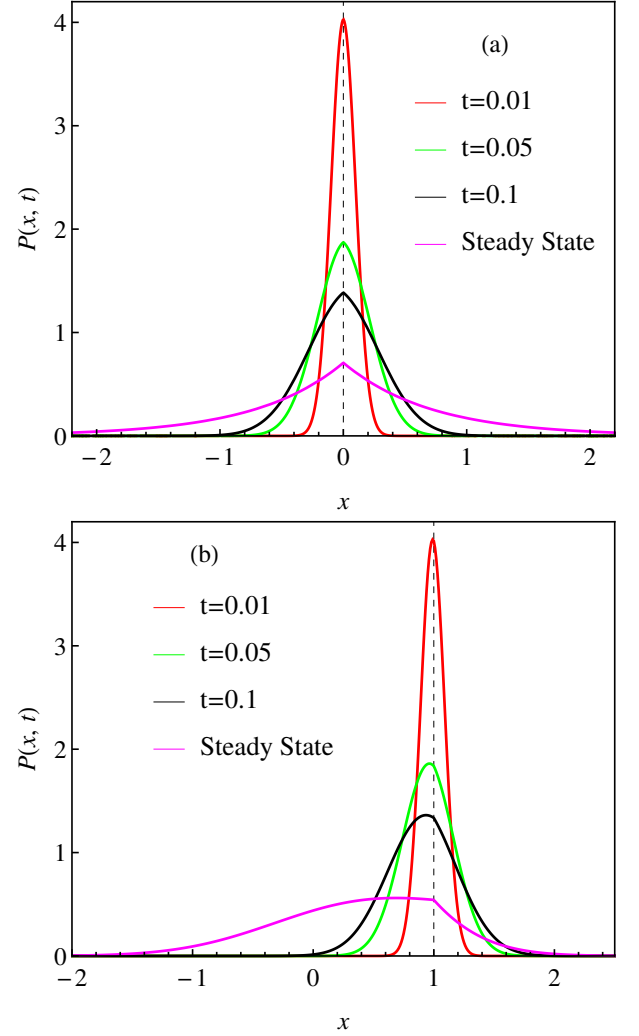


FIG. 5. (Color online) The time dependent propagator for the free and forced diffusion case in the presence of stochastic resetting has been plotted. It depicts the transient of the propagator from time zero to its steady state. The top panel (a) and the bottom panel (b) describe the free and the forced diffusion case respectively. The parameters are chosen to be $r = 0.6$, $\mu = 1.0$ and $D = 0.5$. The vertical dashed line marks the reset position x_0 . In the free case (a) we consider $x_0 = 0$, but for the case (b) we set $x_0 = 1.0$, which is different from the stable minimum $x_{\min} = 0$.

One can generalize resetting to the systems where the resetting takes place to a region instead of a reference point at a constant rate. Also exploring the span or the extremum (namely maximum or minimum) of a dynamics under the resetting paradigm will be very interesting in the connection with the extreme value statistics [20].

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